IV. HAAR WAVELET

In this chapter we will discuss the simplest wavelet in history, the Haar wavelet. First, let us define the concept of wavelet.

Definition 1. *Let $\psi \in L^2(\mathbb{R})$. ψ is called called a (dyadic) orthonormal wavelet in $L^2(\mathbb{R})$, if*

$$\{2^{\frac{n}{2}}\psi(2^n x - l) \mid n, l \in \mathbb{Z}\}$$

is a complete orthonormal system in $L^2(\mathbb{R})$.

For convenience, for each pair of integers n, l, we usually use $\psi_{n,l}(x)$ to denote the function $2^{\frac{n}{2}}\psi(2^n x - l)$. In this course, we only deal with wavelets that are dyadic and orthonormal, and belong to $L^2(\mathbb{R})$, so we often omit *dyadic* and *orthonormal*, and even $L^2(\mathbb{R})$, simply call such functions wavelets. Next we introduce the Haar function:

Definition 2. *Let*

$$H(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

Then $H(x)$ is called the Haar function.

The main goal of this chapter is to prove that the Haar function is a dyadic orthonormal wavelet in $L^2(\mathbb{R})$. Namely

Theorem 1. *Let $H(x)$ be the Haar wavelet. Then $\{2^{\frac{n}{2}}H(2^n x - l) \mid n, l \in \mathbb{Z}\}$ is a complete orthonormal system in $L^2(\mathbb{R})$.*

We prove this theorem through several steps. Let us check first that $\{2^{\frac{n}{2}}H(2^n x - l) \mid n, l \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$. To prove this, let us first write $H_{n,l}(x) = 2^{\frac{n}{2}}H(2^n x - l)$ in an explicit way: For each pair of integers n, l,

$$H_{n,l}(x) = \begin{cases} 2^{\frac{n}{2}} & x \in [\frac{2l}{2^{n+1}}, \frac{2l+1}{2^{n+1}}) \\ -2^{\frac{n}{2}} & x \in [\frac{2l+1}{2^{n+1}}, \frac{2l+2}{2^{n+1}}) \\ 0 & \textit{otherwise.} \end{cases}$$

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For any function $f : \mathbb{R} \longrightarrow \mathbb{C}$, we call the set $\overline{\{x \in \mathbb{R} | |f(x)| \neq 0\}}$ the support of function f. We usually denote this set as $supp(f)$.

Proposition 1. *Let $H(x)$ be the Haar wavelet. Then $\{2^{\frac{n}{2}} H(2^n x - l) \mid n, l \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$.*

Proof. First of all, for any integers n and l,

$$||H_{n,l}||_2^2 = \int_{-\infty}^{\infty} |H_{n,l}(x)|^2 dx = \int_{-\infty}^{\infty} 2^n |H(2^n x - l)|^2 dx.$$

With a change of variable $u = 2^n x - l$, the above integral becomes

$$\int_{-\infty}^{\infty} |H(u)|^2 du = \int_0^1 1 dx = 1.$$

Note that for any integers n and l, $supp(H_{n,l}) = [\frac{l}{2^n}, \frac{l+1}{2^n}]$. So for two functions H_{n,l_1} and H_{n,l_2} with $n, l_1, l_2 \in \mathbb{Z}$ and $l_1 \neq l_2$, we see that the intersection of $supp(H_{n,l_1})$ and $supp(H_{n,l_1})$ is at most a set of single point. Therefore

$$\langle H_{n,l_1}, H_{n,l_2} \rangle = \int_{-\infty}^{\infty} H_{n,l_1}(x) \cdot H_{n,l_2}(x) dx = 0.$$

Now let us look at two functions H_{n_1,l_1} and H_{n_2,l_2} with $n_1, n_2, l_1, l_2 \in \mathbb{Z}$ and $n_2 > n_1$, $l_1 \neq l_2$. We write

$$supp(H_{n_1,l_1}) = [\frac{l_1}{2^{n_1}}, \frac{l_1+1}{2^{n_1}}] = [\frac{2^{n_2-n_1} l_1}{2^{n_2}}, \frac{2^{n_2-n_1}(l_1+1)}{2^{n_2}}],$$

we further write it as the union of sets, namely $supp(H_{n_1,l_1}) =$

$$[\frac{2^{n_2-n_1} l_1}{2^{n_2}}, \frac{2^{n_2-n_1} l_1 + 2^{n_2-n_1-1}}{2^{n_2}}] \cup [\frac{2^{n_2-n_1} l_1 + 2^{n_2-n_1-1}}{2^{n_2}}, \frac{2^{n_2-n_1} l_1 + 2^{n_2-n_1}}{2^{n_2}}],$$

where H_{n_1,l_1} takes value 2^{n_1} on the first set in the union above and -2^{n_1} on the second set in the union above (except an end point). Comparing it with

$$supp(H_{n_2,l_2}) = [\frac{l_2}{2^{n_2}}, \frac{l_2+1}{2^{n_2}}],$$

we see that if the intersection of $supp(H_{n_1,l_1})$ and $supp(H_{n_2,l_2})$ is not empty, or a set of a single point, then $supp(H_{n_2,l_2})$ is contained in one of the sets in the union above where H_{n_1,l_1} is constant. Hence $\langle H_{n_1,l_1}, H_{n_2,l_2} \rangle$ is always 0. This finishes our proof. □

In order to complete the proof of Theorem 1, we only need to prove that the orthonormal system $\{2^{\frac{n}{2}} H(2^n x - l) \mid n, l \in \mathbb{Z}\}$ is complete. According to Theorem 1 in Chapter 1, we only need to prove that for any $f \in L^2(\mathbb{R})$,

$$||f||_2^2 = \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2.$$

Let $V = \{f \in L^2(\mathbb{R})\big|\exists M > 0, \exists m \in \mathbb{Z}$, s.t. $supp(f) \subset [-M,M], f|_{[\frac{j}{2^m},\frac{j+1}{2^m})}$ is constant for any $j \in \mathbb{Z}\}$. We first prove that the above equality holds for the subset V of $L^2(\mathbb{R})$ defined above.

Lemma 1. *Let the subset V of $L^2(\mathbb{R})$ be as defined above. Then for any $f \in V$,*

$$||f||_2^2 = \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2.$$

Proof. To simplify the computation, note that for any $f \in L^2(\mathbb{R})$, if we let

$$f_+(x) = \begin{cases} f(x) & x \in [0,\infty) \\ 0 & otherwise \end{cases}$$

$$f_-(x) = \begin{cases} f(x) & x \in (-\infty,0) \\ 0 & otherwise \end{cases}$$

then $||f||_2^2 = ||f_+||_2^2 + ||f_-||_2^2$. Also for any pair of integers n and l, $supp(H_{n,l})$ is contained either in $[0,\infty)$ or $(-\infty,0]$, so either $\langle f_+, H_{n,l}\rangle = 0$, or $\langle f_-, H_{n,l}\rangle = 0$. Hence

$$|\langle f, H_{n,l}\rangle|^2 = |\langle f_+, H_{n,l}\rangle + \langle f_-, H_{n,l}\rangle|^2 = |\langle f_+, H_{n,l}\rangle|^2 + |\langle f_-, H_{n,l}\rangle|^2.$$

Therefore to prove the lemma, we only need to check whether the identity holds for functions in V whose support is contained either in $[0,\infty)$ or in $(-\infty,0]$. Without loss of generality, let us assume that for some $N \in \mathbb{Z}$ and some $M \in \mathbb{N}$, $supp(f) \subset [0, 2^{M-N}]$. Specifically,

$$f(x) = \begin{cases} A_{N,l} & x \in [\frac{l}{2^N}, \frac{l+1}{2^N}), l \in \{0,1,2,...,2^M-1\} \\ 0 & otherwise \end{cases}$$

where $A_{N,l} \in \mathbb{C}$ for each $l \in \{0,1,2,...,2^M-1\}$. First, we compute to get

$$||f||_2^2 = \frac{1}{2^N}\sum_{l=0}^{2^M-1} |A_{N,l}|^2.$$

It is more complicated to compute $\langle f, H_{n,l}\rangle$. First we note that for any integer $n \geq N$, and any $l \in \mathbb{Z}$, we have $\langle f, H_{n,l}\rangle = 0$ since whenever the intersection of $supp(f)$ and $supp(H_{n,l})$ is other than empty set or a set of single point, $supp(H_{n,l}) =$

$[\frac{l}{2^n}, \frac{l+1}{2^n}]$ is contained in one of the sets of the form $[\frac{l}{2^N}, \frac{l+1}{2^N}]$ with $l \in \{0, 1, 2, ..., 2^M - 1\}$ on which f is constant.

When $n = N-1$, if $l \notin \{0, 1, 2, ..., 2^{M-1} - 1\}$, then $\langle f, H_{N-1,l} \rangle = 0$. Also

$$\langle f, H_{N-1,0} \rangle = 2^{\frac{N-1}{2}} \frac{A_{N,0} - A_{N,1}}{2^N},$$

$$\langle f, H_{N-1,1} \rangle = 2^{\frac{N-1}{2}} \frac{A_{N,2} - A_{N,3}}{2^N},$$

$$...,$$

$$\langle f, H_{N-1,2^{M-1}-1} \rangle = 2^{\frac{N-1}{2}} \frac{A_{N,2^M-2} - A_{N,2^M-1}}{2^N}.$$

When $n = N-2$, if $l \notin \{0, 1, 2, ..., 2^{M-2} - 1\}$, then $\langle f, H_{N-2,l} \rangle = 0$. Also

$$\langle f, H_{N-2,0} \rangle = 2^{\frac{N-2}{2}} \frac{A_{N,0} + A_{N,1} - A_{N,2} - A_{N,3}}{2^N},$$

$$\langle f, H_{N-2,1} \rangle = 2^{\frac{N-1}{2}} \frac{A_{N,4} + A_{N,5} - A_{N,6} - A_{N,7}}{2^N},$$

$$...,$$

$$\langle f, H_{N-2,2^{M-2}-1} \rangle = 2^{\frac{N-1}{2}} \frac{A_{N,2^M-4} + A_{N,2^M-3} - A_{N,2^M-2} - A_{N,2^M-1}}{2^N}.$$

$$......,$$

When $n = N - M$, if $l \neq 0$, then $\langle f, H_{N-M,l} \rangle = 0$. Also

$$\langle f, H_{N-M,0} \rangle = 2^{\frac{N-M}{2}} \frac{A_{N,0} + ... + A_{N,2^{M-1}-1} - A_{N,2^{M-1}} - ... - A_{N,2^M-1}}{2^N}.$$

When $n < N - M$, we let $n = N - M - p$ with $p \in \mathbb{N}$, likewise if $l \neq 0$, then $\langle f, H_{N-M-p,l} \rangle = 0$. Also

$$\langle f, H_{N-M-p,0} \rangle = 2^{\frac{N-M-p}{2}} \frac{A_{N,0} + A_{N,1} + ... + A_{N,2^M-2} + A_{N,2^M-1}}{2^N}.$$

Thus, to check that $||f||_2^2 = \sum_{n,l \in \mathbb{Z}} |\langle f, H_{n,l} \rangle|^2$, we only need to prove the following elementary lemma, which will be left to the reader. □

Lemma 2. *Let $M \in \mathbb{N}$ and $\{A_l\}_{l=0}^{2^M-1} \subset \mathbb{C}$. Then*

$$\sum_{l=0}^{2^M-1} |A_l|^2$$

$$= \frac{|A_0 - A_1|^2 + |A_2 - A_3|^2 + ... + |A_{2^M-2} - A_{2^M-1}|^2}{2}$$

$$+\frac{|A_0+A_1-A_2-A_3|^2+...+|A_{2^M-4}+A_{2^M-3}-A_{2^M-2}-A_{2^M-1}|^2}{4}$$

$$+\frac{|A_0+A_1+A_2+A_3-A_4-A_5-A_6-A_7|^2+.........}{8}$$

$$+.....$$

$$+\frac{|A_0+A_1+A_2+...+A_{2^{M-1}-1}-A_{2^{M-1}}-A_{2^{M-1}+1}-...-A_{2^M-1}|^2}{2^M}$$

$$+\frac{|A_0+A_1+A_2+...+A_{2^M-2}+A_{2^M-1}|^2}{2^M}.$$

To complete the proof of Theorem 1, we will use the fact that the set V defined above is dense in $L^2(\mathbb{R})$ under the $L^2(\mathbb{R})$ norm, the proof of this fact is beyond the scope of this course. To be more specific, we have

Lemma 3. *For any $f \in L^2(\mathbb{R})$, any $\varepsilon > 0$, there is a function $g \in V$, such that $||f-g||_2 < \varepsilon$.*

Proof of Theorem 1. We only need to prove that for any $f \in L^2(\mathbb{R})$,

$$||f||_2^2 = \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2.$$

If $||f||_2 = 0$, then $f = 0$ and there is nothing to prove. If $||f||_2 \neq 0$, we note first that since $\{2^{\frac{n}{2}}H(2^n x - l) \mid n, l \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$, according to Bessel's Inequlity in Chapter 1, $\sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2$ is convergent and therefore well-defined. In fact,

$$\sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2 \leq ||f||_2^2.$$

Now for any $\varepsilon > 0$, take $\varepsilon_1 = \min\{\frac{\varepsilon}{6||f||_2}, ||f||_2\}$, then according to Lemma 3, there is a function $g \in V$, such that $||f-g||_2 < \varepsilon_1$. Thus

$$||g||_2 = ||f-g+g||_2 \leq ||f-g||_2 + ||g||_2 < 2||f||_2.$$

Also, by Lemma 1,

$$||g||_2^2 = \sum_{n,l\in\mathbb{Z}} |\langle g, H_{n,l}\rangle|^2.$$

Hence,

$$\big|||f||_2^2 - \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2\big|$$

$$= \big|||f||_2^2 - ||g||_2^2 + \sum_{n,l\in\mathbb{Z}} |\langle g, H_{n,l}\rangle|^2 - \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2\big|$$

$$\leq \big| ||f||_2^2 - ||g||_2^2 \big| + \big| \sum_{n,l\in\mathbb{Z}} |\langle g, H_{n,l}\rangle|^2 - \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2 \big|.$$

Let us denote the above two terms as I and II respectively. We will treat them separately. For the first term I above, by triangle inequality of the norm $||\cdot||_2$, and the fact that $||g||_2 < 2||f||_2$, we have

$$I = \big| ||f||_2^2 - ||g||_2^2 \big| = \big| ||f||_2 - ||g||_2 \big| \cdot (||f||_2 + ||g||_2)$$

$$\leq ||f-g||_2 \cdot (||f||_2 + ||g||_2) < \varepsilon_1 \cdot 3||f||_2 \leq \frac{\varepsilon}{2}.$$

For the second term, we have

$$II = \big| \sum_{n,l\in\mathbb{Z}} |\langle g, H_{n,l}\rangle|^2 - \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2 \big| = \big| \sum_{n,l\in\mathbb{Z}} (|\langle g, H_{n,l}\rangle|^2 - |\langle f, H_{n,l}\rangle|^2) \big|$$

$$\leq \sum_{n,l\in\mathbb{Z}} \big| |\langle g, H_{n,l}\rangle| - |\langle f, H_{n,l}\rangle| \big| \cdot |\langle g, H_{n,l}\rangle| + \sum_{n,l\in\mathbb{Z}} \big| |\langle g, H_{n,l}\rangle| - |\langle f, H_{n,l}\rangle| \big| \cdot |\langle f, H_{n,l}\rangle|$$

Now by triangle inequality of the norm of complex numbers, Cauchy-Schwartz Inequality of inner product on $l^2(\mathbb{Z})$ and Bessel Inequality, in that order, we see that

$$III \leq \sum_{n,l\in\mathbb{Z}} |\langle g-f, H_{n,l}\rangle| \cdot |\langle g, H_{n,l}\rangle| + \sum_{n,l\in\mathbb{Z}} |\langle g-f, H_{n,l}\rangle| \cdot |\langle f, H_{n,l}\rangle|$$

$$\leq (\sum_{n,l\in\mathbb{Z}} |\langle g-f, H_{n,l}\rangle|^2)^{\frac{1}{2}} \cdot (\sum_{n,l\in\mathbb{Z}} |\langle g, H_{n,l}\rangle|^2)^{\frac{1}{2}}$$

$$+(\sum_{n,l\in\mathbb{Z}} |\langle g-f, H_{n,l}\rangle|^2)^{\frac{1}{2}} \cdot (\sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2)^{\frac{1}{2}}$$

$$\leq ||f-g||_2 \cdot ||g||_2 + ||f-g||_2 \cdot ||f||_2 < \varepsilon_1 \cdot 3||f||_2 \leq \frac{\varepsilon}{2}.$$

In short, for any $\varepsilon > 0$, we have

$$\big| ||f||_2^2 - \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2 \big| < \varepsilon.$$

Thus for any $f \in L^2(\mathbb{R})$, we have

$$||f||_2^2 = \sum_{n,l\in\mathbb{Z}} |\langle f, H_{n,l}\rangle|^2.$$

□